On the Parameter Plane of Nonlinear Coupled Oscillators

Wolfgang Metzler, Achim Brelle, and Klaus-Dieter Schmidt
Fachbereich Mathematik/Informatik, Universität Kassel (GhK), Kassel, FRG

Gerrit Danker, Matthias Köppe, and Daniel Mahrenholz
Klasse 10.2, Landesgymnasium „Werner von Siemens" Magdeburg (Spezialschule mntR), Magdeburg, FRG

Z. Naturforsch. **48a**, 655–662 (1993); received February 16, 1993

Dedicated to the first author's late teacher and friend Friedrich Wille

Two well-known bifurcation routes to chaos of two-dimensional coupled logistic maps are embedded in a two-parameter plane of a canonical nonlinear oscillator which contains a non-analytic analogon to the Mandelbrot set.

1. Introduction

Much insight into the dynamics of nonlinear dynamical systems has been gained through studies of discrete mappings. In the analytical case, especially for the complex logistic map

$$P_c\colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}, \quad z \mapsto z^2 + c \quad (c \in \mathbb{C}), \tag{1}$$

where $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, there exists, as everybody knows, a beautiful mathematical theory which goes back to the famous work of Julia [1] and Fatou [2] and has been continued by Brolin [3], Mandelbrot [4], Douady [5], Douady and Hubbard [6] and many others.

Especially, Douady and Hubbard studied the hyperbolic components of the Mandelbrot set

$$M = \left\{c \in \mathbb{C} \mid P_c^{\circ n}(0) \underset{n \to \infty}{\not\longrightarrow} \infty\right\}. \tag{2}$$

Defining

$$H(M) = \{c \in \mathbb{C} \mid P_c \textit{ has an attracting periodic orbit}\}, \tag{3}$$

any connected component of $H(M)$ is called a hyperbolic component of M. Now, the hyperbolic conjecture for polynomials P_c in degree 2 reads

$$\operatorname{int}(M) = H(M). \tag{HC}$$

(HC) is, as far as we know, yet an open problem. But, nevertheless, it is easy to determine the subsets of M for which P_c has an attracting fixed point or an attracting cycle of period 2 etc. Each of them is simply connected beginning with the main body, being the interior of a cardioid and representing the attracting fixed point, followed by buds becoming smaller and smaller and representing periodic attractors of increasing orders.

Due to the Cauchy-Riemann differential equations, in the analytical case both Lyapunov exponents are identical. Thus, there cannot exist any chaotic attractor. That is, all attractors of an analytic map are periodic. Chaos only appears on Julia sets each of which is the closure of the set of the repelling periodic points.

On the other hand, in the case of non-analytic (here 2D-) maps we of course do have both, periodic and chaotic attractors, dependent on a particular parameter value respectively coexisting. In the present paper, two already known bifurcation routes to chaos of two coupled logistic maps (Kaneko [7], Metzler et al. [8]) are embedded in a two-parameter plane of a canonical nonlinear oscillator which contains a "non-analytic analogon" to the well-known Mandelbrot set (2).

2. A Canonical Two-Parameter Form of Coupled Oscillators

In 1987, Metzler et al. [8] studied a "crosswise" symmetric coupling of two identical logistic maps. Related couplings of nonlinear oscillators have been discussed by Kaneko [7] as well as Hogg and Huberman [9]. They correspondingly found quasiperiodic behavior with frequency lockings as well as bifurcations into

Reprint requests to W. Metzler, Fachbereich Mathematik/Informatik, Universität Kassel (GhK), Heinrich-Plett-Str. 40, W-3500 Kassel, FRG.

0932-0784 / 93 / 0500-0655 $ 01.30/0. – Please order a reprint rather than making your own copy.

aperiodic (chaotic) behavior dependent, in either of the three cases, on only one control parameter. The present work should unify their research by introducing a second control parameter and studying the dynamics of the oscillators in question in a two-dimensional parameter plane just as it is fairly known from the Mandelbrot set in the analytical case.

The coupled logistic map from [8] is given by the equations

$$x_{n+1} = (1+a)\, x_n + a(y_n - x_n^2)\,,$$
$$y_{n+1} = (1+a)\, y_n + a(x_n - y_n^2)\,. \tag{4}$$

Use of transformations

$$x_n = \frac{1}{a} u_n + \frac{1+a}{2a}\,, \quad y_n = \frac{1}{a} v_n + \frac{1+a}{2a}\,, \tag{T1}$$

results in

$$u_{n+1} = \frac{3a^2 + 2a - 1}{4} + a v_n - u_n^2\,,$$
$$v_{n+1} = \frac{3a^2 + 2a - 1}{4} + a u_n - v_n^2\,, \tag{5}$$

respectively, with

$$b = \frac{3a^2 + 2a - 1}{4} \tag{6}$$

in

$$u_{n+1} = b + a v_n - u_n^2\,,$$
$$v_{n+1} = b + a u_n - v_n^2\,. \tag{O}$$

Equations (O) constitute a two-parameter (a, b) canonical form of a nonlinear coupled oscillator.

Another transformation, different from (T1), has been proposed by Danker (one of the young authors). Applying

$$x_n = u_n + \frac{1+a}{2a}\,, \quad y_n = v_n + \frac{1+a}{2a} \tag{T2}$$

to (4) results in

$$u_{n+1} = b + a(v_n - u_n^2)\,,$$
$$v_{n+1} = b + a(u_n - v_n^2) \tag{7}$$

with

$$b = \frac{3a^2 + 2a - 1}{4a}\,. \tag{8}$$

Kaneko's equations read

$$x_{n+1} = 1 - A x_n^2 + D(y_n - x_n)\,,$$
$$y_{n+1} = 1 - A y_n^2 + D(x_n - y_n)\,, \tag{9}$$

where D is an arbitrary but fixed coupling parameter ($D = 0.1$ in [7]) and A ($0 < A < 2$) is the bifurcation parameter. For $A > 1.5$, Kaneko experimentally found hyperchaos, i.e. both Lyapunov exponents are positive. The transformations

$$x_n = \frac{1}{A} u_n - \frac{D}{2A}\,, \quad y_n = \frac{1}{A} v_n - \frac{D}{2A}\,, \tag{T3}$$

applied to (9), yield

$$u_{n+1} = \frac{2D - D^2 + 4A}{4} + D v_n - u_n^2\,,$$
$$v_{n+1} = \frac{2D - D^2 + 4A}{4} + D u_n - v_n^2\,, \tag{10}$$

and this is the canonical form (O) with

$$a = D\,, \quad b = \frac{2D - D^2 + 4A}{4}\,. \tag{11}$$

Obviously, (11) describes a parameter representation of a straight line parallel to the b-axis in the (a, b)-parameter plane of (O), cf. Figure 1. Thus, transition to chaos of Kaneko's oscillator (9) happens on the line given by (11). The parabola in Fig. 1 describes the curve $b = \frac{1}{4}(3a^2 + 2a - 1)$ from (6). Therefore, transition to chaos of (4), which has been discussed in detail in [8], takes place along this parabola. Figure 2a shows some of the bifurcation points and some couples of parameter values (a, b) for which stable attractors of (4) have been observed in [8] (compare Figure 2b). Figure 2c contains four typical examples of attractors of (4) (cf. [10]) which are marked in Figure 2a. In the same way, one can easily localize Kaneko's chaotic attractors from [7] on the straight line in Figure 1. Some of those are hyperchaotic, as will be shown in the following section, too. Therefore, by (6) and (11), respectively, two routes to chaos are embedded in the two-parameter plane of the canonical oscillator (O) (cf. Figure 1).

3. Parameter Plane and Attractors

Figure 1 shows the parameter plane of the oscillator (O). The dark figure looking like a 2D-projection of a

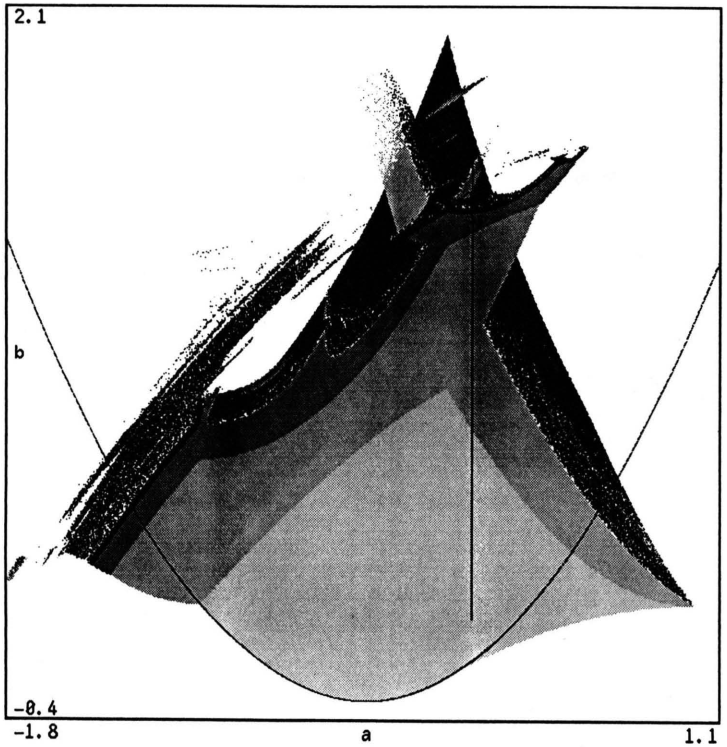
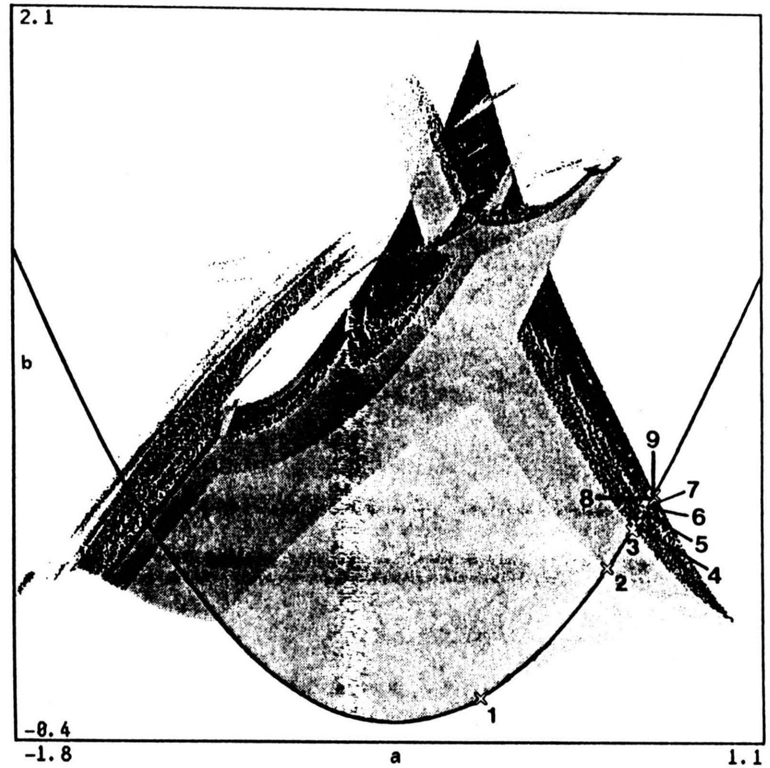


Fig. 1. Mandelbrot set M_O of the oscillator (O) with two routes to chaos, parabola: Metzler et al. [8], straight line: Kaneko [7].

Fig. 2a. The route to chaos of the coupled logistic map (4) in the ▶ (*a*, *b*)-parameter space.



	a	*b*
1:	0	−0.25
2:	0.5	0.1875
3:	0.6	0.32
4:	0.614	0.339747
5:	0.645	0.384519
6:	0.655658	0.400245
7:	0.66	0.4067
8:	0.678	0.433763
9:	0.684	0.442892

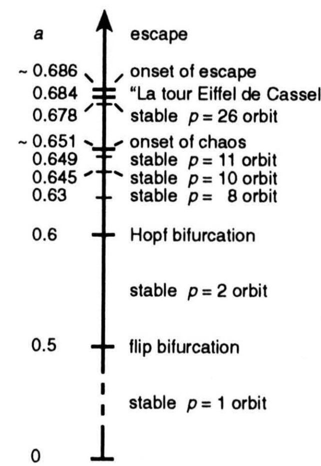


Fig. 2b. Schematic representation of the bifurcation route to chaos of the map (4).

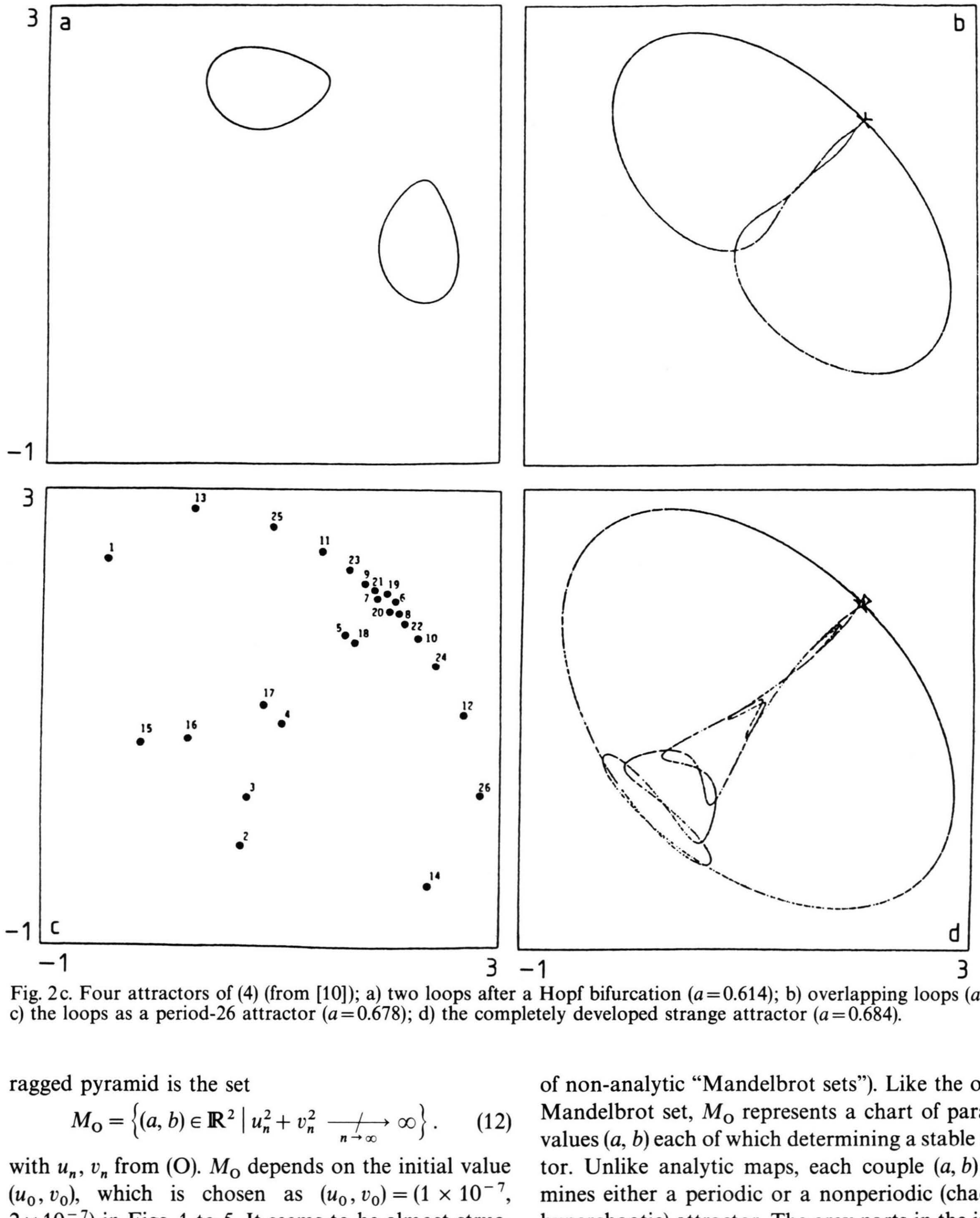


Fig. 2c. Four attractors of (4) (from [10]); a) two loops after a Hopf bifurcation ($a=0.614$); b) overlapping loops ($a=0.66$); c) the loops as a period-26 attractor ($a=0.678$); d) the completely developed strange attractor ($a=0.684$).

ragged pyramid is the set

$$M_O = \left\{(a, b) \in \mathbb{R}^2 \mid u_n^2 + v_n^2 \underset{n \to \infty}{\not\longrightarrow} \infty\right\}. \quad (12)$$

with u_n, v_n from (O). M_O depends on the initial value (u_0, v_0), which is chosen as $(u_0, v_0) = (1 \times 10^{-7}, 2 \times 10^{-7})$ in Figs. 1 to 5. It seems to be almost structurally stable, which means in our present context that small changes in the initial values imply only modest variations of the global structure of M_O.

M_O is a non-analytic analogue to the Mandelbrot set for analytic maps (see also [11] for other examples of non-analytic "Mandelbrot sets"). Like the original Mandelbrot set, M_O represents a chart of parameter values (a, b) each of which determining a stable attractor. Unlike analytic maps, each couple (a, b) determines either a periodic or a nonperiodic (chaotic or hyperchaotic) attractor. The grey parts in the interior of M_O represent stable periodic orbits with low periods (about up to 15). If we approach the boundary of M_O, high order periods are mixing with chaotic and hyperchaotic attractors. This local structure of M_O sensitively depends on the initial state (u_0, v_0) in the

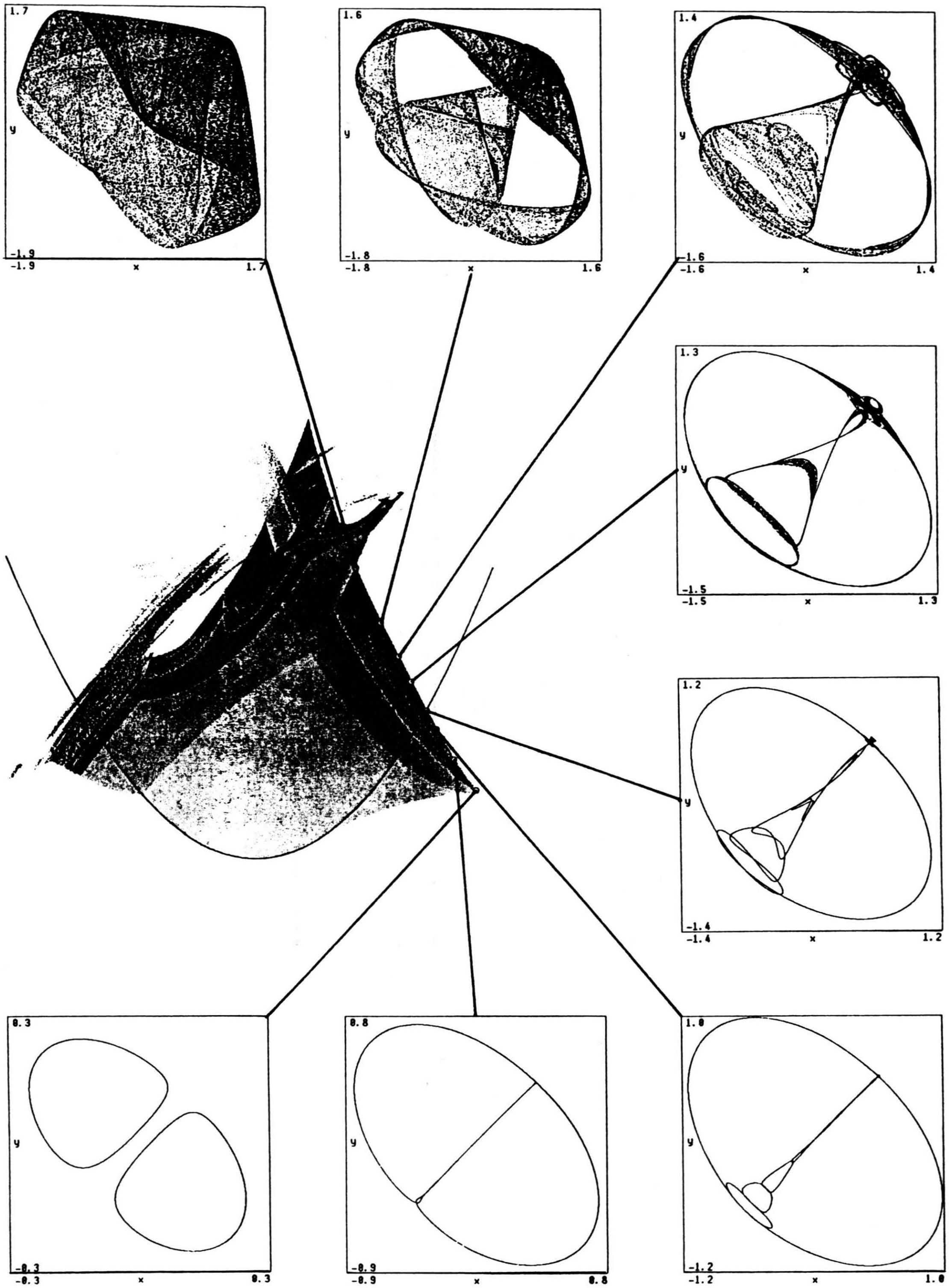


Fig. 3. Selected attractors for parameter values $(a, b) \in M_O$.

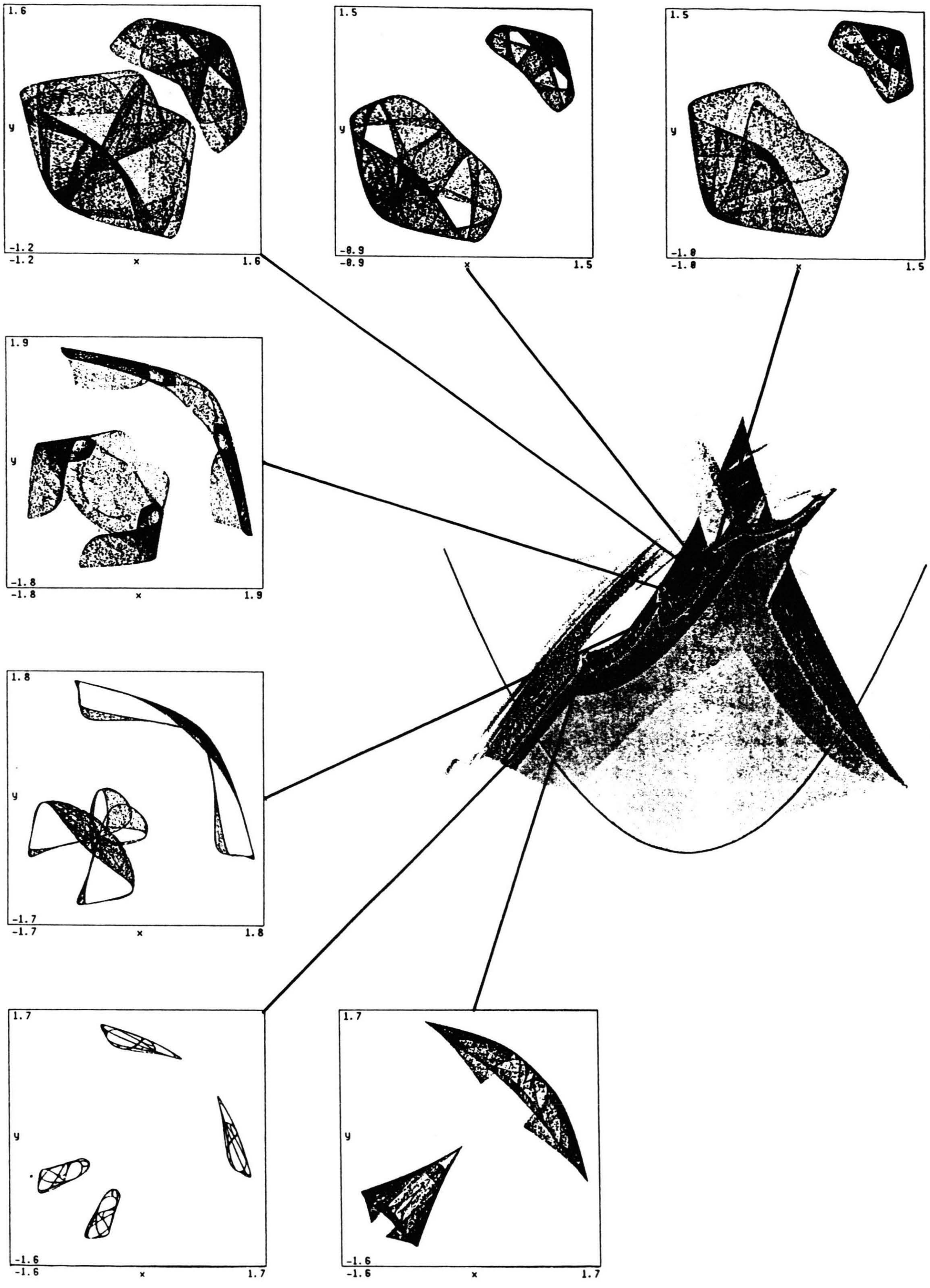


Fig. 4. Like Figure 3.

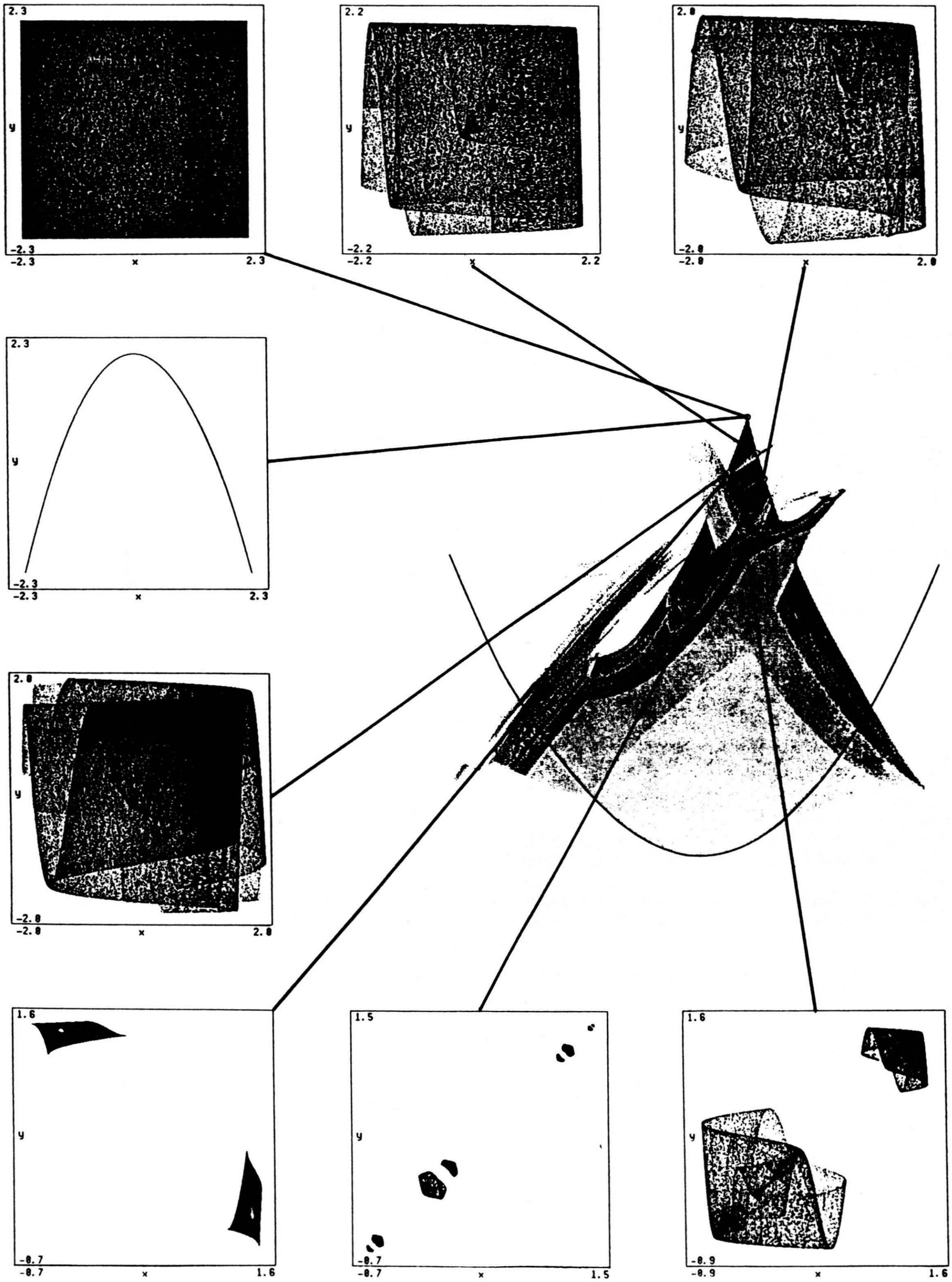


Fig. 5. Like Figs. 3 and 4 with coexisting attractors at the top of M_O, i.e., for $(a, b) = (0, 2)$ (see text).

definition (12) of M_O, e.g. parameter values with periodic attractors can change to aperiodic attractors or vice versa. Figures 3 and 4, respectively, give an insight into the variety to M_O. For most of them, Lyapunov exponents have been calculated, and, of course, the numerical results confirm the pictures.

Figure 5, however, contains a surprising phenomenon, at least at the first glance. For the parameter value $(a, b) = (0, 2)$, that is the top of M_O, dependent on the initial values numerous different but simple attractors are coexisting. Two of them are the filled-in square and the parabola on top of the left row in Figure 5. Obviously, the reason is, that Eqs. (O) are decoupled in this case ($a=0$). This phenomenon will be disentangled in detail at another place [12]. To be correct, we must add that in Fig. 5 all attractors have been calculated with the same initial value (1×10^{-7}, 2×10^{-7}) and the filled-in square results from $(a, b) = (0, 1.999)$.

4. Outlook

The transformation of a one-parameter-dependent nonlinear oscillator into a two-parameter map (O) was stimulated by some analytical work about the Hénon map [13] by Cvitanović and others [14]. It is praiseworthy that the three schoolboys from Magdeburg have done the first concrete step in that direction. The representation (O) of two-dimensional coupled logistic maps and more general of nonlinear oscillators opens the possibility to analyze attractors and bifurcation routes in a (two-parameter) Mandelbrot set. Further, there exists a conjecture about the Hénon attractor according to which it should be the closure of the instable manifold of a specific fixed point [15]. Such a result should be interesting in our context, too.

Acknowledgement

Paper presented at the 3rd Annual Meeting of ENGADYN, Grenoble, 1992. Thanks to all ENGADYN participants for engaged discussions. We further appreciate M. Klein's spontaneous help in computing the Lyapunov spectra.

[1] G. Julia, J. Math. Pures Appl. **8**, 47 (1918).
[2] P. Fatou, Bull. Soc. Math. France **47**, 161 (1919); **48**, 33, 208 (1920).
[3] H. Brolin, Ark. Mat. **6**, 103 (1965).
[4] B. B. Mandelbrot, in: Lecture Notes in Mathematics, Vol. 615 (P. Bernhard and T. Ratin, eds.), Springer, New York 1977, p. 83.
[5] A. Douady, Systemes dynamiques holomorphes, Sém. Bourbaki, 35e anné, No. 599; Astérisque **105–106**, 39 (1983).
[6] A. Douady and J. H. Hubbard, Étude dynamique des polynômes complexes, I, II, Publ. Math. Orsay 1984, 1985.
[7] K. Kaneko, Prog. Theor. Phys. **69**, 1427 (1983).
[8] W. Metzler, W. Beau, W. Frees, and A. Ueberla, Z. Naturforsch. **42a**, 310 (1987).
[9] T. Hogg and B. A. Huberman, Phys. Rev. **A 29**, 275 (1984).
[10] W. Beau, W. H. Hehl, and W. Metzler, Informatik Forsch. Entw. **2**, 122 (1987).
[11] W. Metzler, A. Brelle, and K.-D. Schmidt, J. Bifurc. and Chaos **2**, 241 (1992).
[12] W. Metzler, K.-D. Schmidt, and R. Pilgram, Mandelbrot Set for Nonlinear Coupled Oscillators (to appear).
[13] M. Hénon, Commun. Math. Phys. **50**, 69 (1976).
[14] P. Cvitanović, H. Gunarathne, and J. Procaccia, Phys. Rev. **A 38**, 1503 (1988).
[15] D. Ruelle, Chaotic Evolution and Strange Attractors, Cambridge Univ. Press, Cambridge 1989.